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A solving tool for fuzzy quadratic optimal control problems

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Abstract. In this paper we propose an iterative method to solve an optimal control problem, with fuzzy target and constraints. The algorithm is developed in such a way as to satisfy the target function and the constraints. The algorithm can be applied only if a method exists to solve a crisp parametric sub-problem obtained by the original one. This is the case for a quadratic-linear target function with linear constraints, for which some well established solvable methods exist for the crisp associated sub-problem. A numerical test confirmed the good convergence properties.

Keywords: fuzzy, mathematical programming.

JEL Classification Numbers: C6.

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1 Introduction

This paper deals with fuzzy optimal control problem (FOCP). Given the theoretical difficulties that arise in the solution methods for optimal control problems, few fuzzy extensions were proposed for such cases. We mention only few contribution in the field of optimal control and dynamic programming problems; see [5], [9], [11], [13], [14], [15], [17].

In vagueness problems, *fuzzy* goals and constraints are considered, and usually a *satisfaction* degree is obtained both for the target function and for the constraints, see [1]. On the other side, in ambiguity problems some of the coefficients are *fuzzy numbers*. Some example of the latter case can be found in [8] and [12]. In this paper we shall focus the attention on problems with vagueness (*possibilistic* optimization problems) where it is required that both the target function and the constraints satisfy as much as possible some required performances, represented by fuzzy numbers, each of them defined by a suitable *membership functions*. Subsequently, the values of the membership functions are aggregated by the *minimum* triangular norm (*t-norm* MIN), obtaining the best *compromise* solution. For compromise problems and their solution, refer to the surveys presented in [8].

In what follows, an iterative algorithm for fuzzy optimization problems is proposed and applied to a particular case of optimal control problem. The original problem is decomposed into a set of crisp sub-problems, each of them depending on a parameter and subsequently solved. The value of the parameter is adjusted at each iteration, in such a way as to converge to the optimal solution. The approach requires a solving method for the related parametric crisp sub-problems. It can be found that the algorithm converges to the optimal solution in a finite number of steps.

The paper is organized as follows. Section 2 describes the fuzzy optimization problem. The algorithm is described in Section 3 together with some theoretical result. Section 4 consider the quadratic-linear optimal control problem. Finally, in Section 5 a test simulation is proposed, showing the quick convergence of the proposed algorithm to the optimal compromise solution.

2 Fuzzy optimization problem

Consider the following mathematical programming problem:

$$\begin{cases} \min_{x \in X} f(x) \\ g_i(x) \geq 0 \end{cases} \quad (1)$$

with $x \in R^n$, $g_i \in R^m$, $X \subseteq R^n$, and $f(x)$ is convex function. Let U , the admissible region of problem (1), be a convex set. The problem (1) can be extended to a *possibilistic* optimization problem, where the borders that differentiate satisfactory from unsatisfactory regions are not *rigid* thresholds, but are represented by suitable fuzzy numbers. To this aim, the objective function and the constraints need to be intended in fuzzy sense, as in [20]. That is, an optimizing solution has to satisfy as most as possible both the target and the constraints, namely to maximize the *minimum* degrees of the target function and the ones of all the constraints.

The fuzzy version of the FMP (Fuzzy Mathematical Programming problem), see (1), can be written as:

$$\begin{cases} \widetilde{\min}_{x \in X} f(x) \\ g_i(x) \widetilde{\geq} 0 \end{cases} \quad (2)$$

Let the membership functions $\mu_0(z)$, $\mu_i(z)$ represent the satisfaction degrees of the target and of the constraints respectively. The fuzzy mathematical problem (2) can be converted into the *crisp* non linear problem:

$$\max_{x \in X} C(x) \quad (3)$$

where $C(x)$ represents the global satisfaction degree:

$$C(x) = \min\{\nu_0(x), \nu_1(x), \dots, \nu_m(x)\} \quad (4)$$

and:

$$\nu_0(x) = \mu_0(f(x)), \nu_i(x) = \mu_i(g(x)) \quad (5)$$

for $i = 1, \dots, m$. This problem is equivalent to the following one, in the space R^{n+1} , see [?]:

$$\begin{cases} \max_{x \in X, \lambda \in [0,1]} \lambda \\ \nu_i(x) \geq \lambda, i = 0, \dots, m \end{cases} \quad (6)$$

The satisfaction degrees assigned to each constraint and to the target function are usually represented by continuous and (almost everywhere) differentiable monotonic fuzzy numbers. In particular, $\mu_i(z) : R \rightarrow [0, 1]$, for $i = 1, \dots, m$ are increasing functions, and $\mu_0(z) : R \rightarrow [0, 1]$ is a decreasing function. The following piecewise linear functions are two of the most commonly used for monotonic membership functions (*S*-type and *Z*-type fuzzy numbers respectively):

$$\mu_0(z) = \begin{cases} 1, & z \leq p_0 \\ \frac{z-c_0}{p_0-c_0}, & p_0 < z \leq c_0 \\ 0, & z > c_0 \end{cases} \quad (7)$$

$$\mu_i(z) = \begin{cases} 0, & z \leq p_i \\ \frac{z-p_i}{c_i-p_i}, & p_i < z \leq c_i \\ 1, & z > c_i \end{cases} \quad (8)$$

with $p_i < c_i, i = 1, \dots, m$. Many methods were proposed to solve the problem (6); see the quoted references.

In this paper an iterative algorithm is presented, partially following the approaches proposed by [19] and [16] to the linear programming problem. The method is based on an iterative algorithm, that requires to compute the solution of a parametric crisp sub-problem. The algorithm will be presented in the general case in the following Section 3.

In the Section 4 the procedure will be applied to the quadratic-linear optimal control problem. A numerical example is presented in the final Section 5.

3 An iterative algorithm for FMP problems

In most cases, the non linear problem (6) can be difficult to solve. To this aim, an alternative method is proposed, based on an iterative procedure, under the hypothesis that an associated crisp sub-problem can be solved.

First of all, consider the following parametric problem $P_\lambda, \forall \lambda \in [0, 1]$:

$$\begin{cases} \min_{x \in X} f(x) \\ \nu_i(x) \geq \lambda, i = 1, \dots, m \end{cases} \quad (9)$$

Let be Ω_λ the admissible region of the problem (9) and x_λ the solution of the problem P_λ . Note that the admissible region of the problem (9) is included in the admissible region of the following unconstrained problem P_f (*unconstrained* in the sense that the parametric constraints $\nu_i(x) \geq \lambda$ are not included):

$$\max_{x \in X} \nu_o(x) \quad (10)$$

If (10) has a solution x_f such that $\nu_o(x_f) \leq \min_i \{\nu_i(x_f)\}$, then such solution cannot be ameliorated and $C(x_f) = \nu_o(x_f)$.

Note that the original FMP problem loses its interest if the problem P_0 does not admit the global minimum, as showed by the following example. As a matter of fact, consider the FMP problem:

$$\begin{cases} \min_x \frac{1}{x} \\ x \geq 3 \end{cases} \quad (11)$$

where:

$$\mu_0(z) = e^{-z^2} \quad (12)$$

and:

$$\mu_1(z) = \begin{cases} 1, & 3 \leq z \\ z - 2, & 2 \leq z < 3 \\ 0, & z < 2 \end{cases} \quad (13)$$

We can write:

$$\nu_0(x) = e^{-\frac{1}{x^2}} \quad (14)$$

$$\nu_1(x) = \begin{cases} 1, & 3 \leq x \\ x - 2, & 2 \leq x < 3 \\ 0, & x < 2 \end{cases} \quad (15)$$

In this case, both the FMP problem (11) and its unconstrained related sub-problem (10) have no solution, as it can be easily checked. Really, the problem P_λ has no solution, $\forall \lambda \in [0, 1]$. As a matter of fact, the satisfaction degree $C(x) = \min\{\nu_0(x), \nu_1(x)\}$ becomes, for $x \geq 3$: $C(x) = \nu_0(x) = e^{-\frac{1}{x^2}}$, since $\nu_1(x) = 1, \forall x \geq 3$, and its minimum does not exist.

However suppose that the membership function (12) is changed as follows:

$$\mu_0(z) = \begin{cases} 1, & z < 2 \\ \frac{4-z}{2}, & 2 \leq z < 4 \\ 0, & z \geq 4 \end{cases} \quad (16)$$

the optimal solution now exists, and it is given by all the points of the unbounded interval $[3, +\infty)$.

To avoid similar meaningless cases, we suppose that $\forall i = 0, \dots, m, \exists c_i$ such that $\mu_i(z) = 1, \forall z \geq c_i$ or $\forall z \leq c_i$; this implies that the target function and each constraints are completely satisfied if a threshold is reached, as in (7), (8). Moreover, we require that $\{x \in X : \nu_0(x) = 1\} \neq \emptyset$, and that $\{x \in X : \min_i \{\nu_i(x)\}\} \neq \emptyset$. From a practical point of view, those hypotheses are not serious limitations.

Let $I_\lambda(f)$ be the *level set* of a given function $f(x) : R^n \rightarrow R, x \in X$:

$$I_\lambda(f) = \{x \in R^n : f(x) \geq \lambda\} \quad (17)$$

We can now enunciate the following Propositions 1 and 2.

PROPOSITION 1. For each $\lambda \in [0, 1]$, $\lambda_1 \geq \lambda_2$ implies $\Omega_{\lambda_1} \subseteq \Omega_{\lambda_2}$.

PROOF: Given the hypotheses stated about $g_i(x), f(x)$, since $\mu_i(z) : R \rightarrow [0, 1]$ is continuous and monotonic increasing, it follows that the set Ω_λ is a convex set. Moreover, the sets $I_\lambda(\mu_i), i = 1, \dots, m$, are convex subsets of R , and given the monotonicity of $\mu_i(z)$, $\inf[I_{\lambda_1}(\mu_i)] > \inf[I_{\lambda_2}(\mu_i)], \forall \lambda_1 \geq \lambda_2$. Being $\sup[I_\lambda(\mu_i)] = +\infty, \forall i$, we can conclude that $I_{\lambda_1}(\mu_i) \subseteq I_{\lambda_2}(\mu_i)$. Since X a convex set, $\Omega_\lambda = (\cap_i I_\lambda(\mu_i(g(x)))) \cap X$ is a convex set.

PROPOSITION 2 (Necessary optimality condition). The problem (6) admits a global optimal solution \bar{x} , in correspondence to a value $\bar{\lambda} \in [0, 1]$. Furthermore, $\nu_i(\bar{x}) \geq \bar{\lambda}, \forall i = 0, 1, \dots, m$, and $\exists i \in \{1, \dots, m\} : \nu_i(\bar{x}) = \nu_0(\bar{x}) = \bar{\lambda}$.

PROOF: The function $f(x)$ is a convex function defined over the convex set Ω_λ , and $\nu_0(x_f) \geq \min_i \{\nu_i(x_f)\}$, where $\nu_0(x_0)$ is the solution of the problem P_f . Suppose by contradiction that the optimal solution is in the point x^* and $\nu_0(x^*) \neq \min_i \nu_i(x^*)$. Let us distinguish the following two cases:

- 1) $\nu_0(x^*) > \min_i \nu_i(x^*)$
- 2) $\nu_0(x^*) < \min_i \nu_i(x^*)$

Both $\mu_i(z)$ and $g_i(x)$ are continuous functions. Then $\nu_i(x)$ are continuous, because compounded functions of continuous functions. Consequently, also $\bar{\nu}(x)$ is a continuous function. Then a neighborhood $T_\epsilon(x^*) \subset R^n$ exists so that $\bar{\nu}(x^*) > \nu_0(x^*) = \lambda_0, \forall x \in T_\epsilon(x^*)$. Since f is a convex function, necessarily $x^* \neq x_0$ because x_0 is the unique minimizing point of f . It follows that $\exists v_{x^*} \in R^n$, depending by x^* with $\|v_{x^*}\| = 1$, so that the function $G(\delta) = f(x_0 + v_{x^*})$ is locally monotonic in $\delta = 0$. That is, there exists a neighborhood $U(\delta) \subset R$ so that, for $\delta > 0$: $G(\delta) > 0$ or $G(\delta) < 0$. It follows that, in all the points of $T_\epsilon(x^*) \cap G(\delta)$, with $\delta > 0$ in the first case, or with $\delta < 0$ in the second case, it holds: $\nu_0(x) > \lambda_0$ and $\bar{\nu}(x) > \lambda_0$. Finally, in all such points: $\min\{\nu_0(x), \bar{\nu}(x)\} > \min\{\nu_0(x^*), \bar{\nu}(x^*)\} = \lambda_0$. Consequently, x^* cannot be the optimizing point, as hypothesized.

The second case, $\nu_0(x^*) < \min_i \nu_i(x^*)$ can be treated in the same way.

Note that the condition stated by the above Proposition 2 is only necessary. In fact, it is very easy to define a function that in every point of its domain satisfies such a condition. For instance, referring to (7), (8), in R^1 and with $m = 1$, the function: $f(x) = \frac{p_0 - c_0}{c_1 - p_1}(x - p_1) + c_0$, for $p_1 \leq x \leq c_1$, satisfies $\nu_0(x) = \nu_1(x), \forall x \in X$, even if all those points are not minimizer ones.

The sufficient condition implies the Pareto optimality for each admissible direction. If we define $J(\lambda) = \{j \in 1, \dots, m : \nu_j(\bar{x}) = \lambda\}$, the sufficiency condition requires that $\forall v \in R^n$, with $\|v\| = 1, \exists \epsilon > 0$ so that, if $(\bar{x} + \delta v) \in X$ with $0 \leq \delta \leq \epsilon$, at least one of the following two conditions be satisfied:

a) $f(\bar{x} + \delta v) \geq f(\bar{x})$

or:

b) $\exists i \in J(\lambda) : \nu_i(\bar{x} + \delta v) \leq \nu_i(\bar{x})$.

Obviously, the condition a) implies $\nu_0(\bar{x} + \delta v) \leq \nu_0(\bar{x})$. Moreover, if $\mu_i(z), \mu_0(z), f(x), g_i(x)$ are differentiable in \bar{x} , from the formulated hypotheses it follows $\mu'_0 \leq 0$ and $\mu'_i \geq 0$, and the two above conditions become:

a.1) $\frac{\partial \nu_0(\bar{x})}{\partial v} \geq 0$, that is $v^T \cdot \nabla \nu_0(\bar{x}) = \mu'_0[f(\bar{x})] \cdot v^T \cdot \nabla f(\bar{x}) \geq 0$, from which $v^T \cdot \nabla f(\bar{x}) \leq 0$

a.2) $\exists i \in J(\lambda) : \frac{\partial \nu_i(\bar{x})}{\partial v} \leq 0$, that is $v^T \cdot \nabla \nu_i(\bar{x}) = \mu'_i[g_i(\bar{x})] \cdot v^T \cdot \nabla g_i(\bar{x}) \leq 0$, from which $v^T \cdot \nabla \nu_i(g_i(\bar{x})) \leq 0$.

The optimization algorithm is based on Propositions 1 and 2. First of all, suppose that an algorithm exist to solve the parametric problem $P_\lambda, \forall \lambda \in [0, 1]$. Let x_f, x_λ be the values of the solution of the unconstrained problems P_f , (10), and of the sub-problem P_λ , (9), respectively. The algorithm modifies at each iteration the value of λ in such a way as to increase the value of the satisfaction degree.

Then, if the hypotheses of Proposition 2 are satisfied, the following bisection algorithm can be applied to solve the FMP problem.

BISECTION ALGORITHM

a) solve the unconstrained problem $P_f: x_f = \operatorname{argmax}_x \nu_0(x)$ (given the stated hypotheses, the existence of a solution is guaranteed). Being $\nu_0(x_f) = 1$ by hypothesis, compute the value $\bar{\nu}(x_f) = \min\{\nu_1(x_f), \dots, \nu_m(x_f)\}$; if $\nu_0(x_f) \leq \bar{\nu}(x_f)$ then stop, and the solution is optimal with satisfaction degree $C(x_0) = \nu_0(x_f) = 1$, and cannot be ameliorated; otherwise ($\bar{\nu}(x_f) < \nu_0(x_f)$), set $\lambda_{inf} = \bar{\nu}(x_f), \lambda_{sup} = \nu_0(x_f), \lambda = \frac{\lambda_{inf} + \lambda_{sup}}{2}$

b) solve the parametric problem P_λ , (9), and compute the values $\nu_0(x_\lambda), \bar{\nu}(x_\lambda) = \min\{\nu_1(x_\lambda), \dots, \nu_m(x_\lambda)\}$. If $|\nu_0(x_\lambda) - \bar{\nu}(x_\lambda)| < \epsilon$ then stop; the optimal solution is reached, with $x^* = x_\lambda$ and satisfaction degree $C(x^*) = \nu_0(x^*) \cong \bar{\nu}(x^*)$, and $\lambda^* = \lambda$. Else:

c) if $\nu_0(x_f) > \bar{\nu}(x_f)$ then set $\lambda \leftarrow \frac{\lambda + \lambda_{sup}}{2}, \lambda_{inf} = \lambda$, goto b). Else ($\nu_0(x_f) < \bar{\nu}(x_f)$):

d) set $\lambda \leftarrow \frac{\lambda + \lambda_{inf}}{2}, \lambda_{sup} = \lambda$, goto b).

Some remarks are in order:

i) The condition $|\nu_0(x_\lambda) - \bar{\nu}(x_\lambda)| < \epsilon$ checks for the equality of the satisfaction degrees for target and constraints, see Proposition 2. As usual, $\epsilon > 0$ is a positive threshold.

ii) Naturally, if $\lambda^* = 0$, it means that the admissible region of problem (6) is empty.

iii) The algorithm implements a simple dichotomic approach. To avoid undesired instability some checks are necessary, and this justifies the use of λ_{inf} , λ_{sup} which represents at each iteration the minimum and the maximum value respectively for the satisfaction degree of the constraints. The value of λ for the next iteration cannot be less than λ_{inf} , neither greater than λ_{sup} . This can happen if at the actual iteration the membership degree of the target suddenly decreased too much, and the simple dichotomic method in this case can produce instability computing a value of λ less than λ_{inf} , or greater than λ_{sup} . In this case, see steps c) and d), the value of λ is forced to an intermediate value among λ_{min} or λ_{sup} and the actual value of λ (usually equal to the satisfaction degree of the constraints, that is $\bar{\nu}(x_\lambda)$).

iv) Propositions 1 and 2 ensure that the algorithm converges in a finite number of steps. In fact, updating the value of λ as in step d), the convergence of the algorithm is assured in at most $\log_2 \frac{1}{\epsilon}$ steps¹. Anyway, a more sophisticated algorithm can improve the speed of convergence, see [6], but they are beyond the scope of this paper.

v) Given the hypotheses, in the optimizing point the sufficient condition is ensured, because starting from x_f the algorithm moves toward the (unique) global optimizing point.

4 The quadratic-linear FOCP problem

The proposed algorithm could be used to solve both linear and non linear FMP and FOCP problems. In the linear case of case many other algorithms exist, see for instance [2], [10], [18] and the references therein. On the other side, the optimization problem (9) can be difficult to be solved in the non linear case, and the previous algorithm cannot be applied. In some particular cases the crisp parametric sub-problem is a standard programming problem, whose solution can be easily obtained, and then the proposed iterative algorithm can be applied.

This is the case of the quadratic FMP and FOCP problems, say QFMC and QFOCP respectively, both of them widely used in the real world applications.

In this Section, the iterative algorithm developed for the general case in the previous Section will be particularized to the QFOCP case. The dynamics is represented by a linear crisp equation (with fixed initial state), while the linear constraints are the final condition are represented by fuzzy constraints. Then the minimum energy QFOCP (2) can be written as:

$$\begin{cases} \min_{u_0, \dots, u_{T-1}} J(u_0, \dots, u_{T-1}) = \frac{1}{2} \sum_{t=1}^{T-1} u_t' Q_t u_t & \text{dynamic equation, } t = 0, \dots, T-1 \\ x_{t+1} = A_t x_t + B_t u_t & \\ x_o = x^0 & \text{initial condition (fixed)} \\ x_T \tilde{\geq} \bar{x} & \text{final state condition} \\ G_t x_t + H_t u_t \tilde{\geq} b_t & \text{linear constraints, } t = 1, \dots, T-1 \end{cases} \quad (18)$$

¹ Of course a first iteration is necessary to solve the unconstrained problem P_f .

with $x_0, x_t, x_T \in R^n$, $u_t \in R^m$, $Q_t \in R^{m \times m}$, $A_t \in R^{n \times n}$, $B_t \in R^{n \times m}$, $G_t \in R^{r \times n}$, $H_t \in R^{r \times m}$, $b_t \in R^r$.

Moreover, let be assigned the membership functions $\mu_0(z)$, $\mu_T(z)$, $\mu_t(z)$. From (5) it follows that:

$$\begin{cases} \nu_0(u_0, \dots, u_{T-1}) = \mu_0(J(u_0, \dots, u_{T-1})) \\ \nu_T(x_T) = \mu_T(x_T) \\ \nu_t(x_t, u_t) = \mu_t(G_t x_t + H_t u_t - b_t). \end{cases} \quad (19)$$

We suppose satisfied the convexity conditions for the target function. Then, disregarding the linear constraints, a global minimum can be obtained for the crisp problem, using the feedback-feedforward method based on the Riccati equation. Let us now consider the fuzzy problem. The region U is a convex set. From the Propositions 1, 2, and the convexity of the target function, an optimal solution exists and the proposed algorithm can be used. The crisp parametric sub-problem (9) becomes a quadratic-linear problem with linear constraints, and can be easily solved using the same standard techniques based on the Riccati equation.

To this purpose, we suggest, among other methods, the so called *penalty function*, see [6], a method which requires an inner loop to reach the optimal solution in the steps a), b) of the QFMP iterative algorithm.

Anywise, for what above said, the sub-problem (9) becomes for the quadratic-linear case (18):

$$\begin{cases} \min \frac{1}{2} \sum_{t=1}^{T-1} u_t' Q_t u_t \\ x_{t+1} = A_t x_t + B_t u_t, \quad x_o = x^0 \\ \nu_T(x_T, u_t) \geq \lambda \\ \nu_t(x_T) \geq \lambda \end{cases} \quad (20)$$

where $\nu_t(x_t, u_t) = \mu_t(G_t x_t + H_t u_t - b_t \geq 0)$, $\nu_T(x_T) = \mu_T(x_T \geq \bar{x})$.

Since $\mu_t(z)$, $\mu_T(z)$ are increasing S -type fuzzy number, the constraints $\nu_t(x_t, u_t) = \mu_t(G_t x_t + H_t u_t - b_t)$ and $\nu_T(x_T, u_t) = \mu_T(x_T)$ can be written as $G_t x_t + H_t u_t - b_t \geq \inf I_\lambda(\mu_t)$ and $x_T \geq \inf I_\lambda(\mu_T)$. Thus the problem P_λ , can be written as the following QFMP with linear constraints:

$$\begin{cases} \min \frac{1}{2} \sum_{t=1}^{T-1} u_t' Q_t u_t \\ x_{t+1} = A_t x_t + B_t u_t, \quad x_o = x^0 \\ G_t x_t + H_t u_t - b_t \geq \inf I_\lambda(\mu_t) \\ x_T \geq \inf I_\lambda(\mu_T) \end{cases} \quad (21)$$

Being satisfied the hypotheses of the Proposition 2, an optimal solution surely exists.

Note that even if the minimum energy QFOCP, the extension to more general type of QFOCP is straightforward.

5 A numerical test for the minimum energy linear QFOCP

The proposed algorithm was tested using the following time-invariant minimum energy QFOCP, with $T = 2$:

$$\begin{cases} \widetilde{\min} \frac{1}{2}(u_o^2 + u_1^2) \\ x_{t+1} = 0.5x_t + 2u_t, x_0 = 4 \\ x_T \widetilde{\geq} 9 \\ u_t \widetilde{\leq} 2, \end{cases} \quad t = 1, 2 \quad (22)$$

where the membership functions of the target function and of the constraints (final state and control variables) are given by:

a) target function:

$$\mu_0(z) = \begin{cases} 1, & z \leq 4 \\ -0.5z + 3, & 4 < z \leq 6 \\ 0, & z > 6 \end{cases} \quad (23)$$

b) final state constraint:

$$\mu_T(z) = \begin{cases} 0, & z \leq 6 \\ \frac{1}{3}z - 2, & 6 < z \leq 9 \\ 1, & z > 9 \end{cases} \quad (24)$$

c) control variables constraints, equal for both the two control variables (so they are not indicized with the subscript t) as in (18)):

$$\mu(u_t) = \begin{cases} 1, & u_t \leq 2 \\ -0.5u_t + 2, & 2 < u_t \leq 4 \\ 0, & u_t > 4 \end{cases} \quad (25)$$

with $t = 1, 2$.

Being $z = u_o^2 + u_1^2$, it is $\nu_0(u_0, u_1) = \mu_o(u_o^2 + u_1^2)$, moreover $\nu_T(x_T) = \mu_T(x_T)$, $\nu(u_0) = \mu(u_o)$, $\nu(u_1) = \mu(u_1)$.

The target function is convex, and the problem verifies all the other hypotheses formulated in the previous Sections to guarantee the applicability and the convergence.

At the iteration n. 0, the problem $P_f, \min_{\frac{1}{2}}\{u_o^2 + u_1^2\}$ with optimal solution $u_0 = u_1 = 0$, and $z = 0$ gives $\nu_0(u_0, u_1) = \nu(u_0) = \nu(u_1) = 1$ and $\nu_{x_T} = 1$. Then $\bar{\nu}(u_0, u_1) = 0$ and $\lambda_{inf} = 0$, $\lambda_{sup} = 1$, thus $\lambda_1 = \frac{\lambda_{inf} + \lambda_{sup}}{2} = 0.5$, see the first row in Table 1. From (21), (22), and from (23), (24), (25) the problem P_λ can be formulated as:

$$\begin{cases} \min_{\frac{1}{2}}(u_o^2 + u_1^2) \\ x_{t+1} = 0.5x_t + 2u_t, x_0 = 4 \\ u_t \leq -2\lambda + 4 \\ x_2 \geq 6\lambda + 6 \end{cases} \quad (26)$$

For the subsequent iteration n.1, with $\lambda = 0.5$, the optimal solution is given by $u_0 = \frac{6}{5}, u_1 = \frac{12}{5}$ with $z = \frac{18}{5}$. Then, being $\nu_0(u_0, u_1) > \bar{\nu}(u_0, u_1)$, it follows that $\lambda_2 = \frac{\lambda_1 + \lambda_{sup}}{2} = \frac{0.5+1}{2} = \frac{3}{4}$, see the second row in Table 1. The updated value of λ is now used for the iteration n.2, and so on, as presented in each row of the Table 1.

With the value $\epsilon = 0.01$, the algorithm converges to the solution in six steps (software package: MatLab).

The obtained results are presented in Table 1, where each rows corresponds to a complete iteration of the procedure. The first column reports the iteration counter, the second and the third ones contains the value of the minimizing solution, u_0, u_1 . The fourth column reports the value of the target function at time t , while the following four columns report the membership degrees of the the three constraints and of the objective function. The column 8 reports the minimum of the membership degrees of the constraints, the column 9 and 10 the values of λ_{inf} , λ_{sup} , finally the last column reports the value of λ which will be applied in the next iteration, λ_{t+1} .

Table 1. Results of the bisection algorithm

t	λ_t	u_0	u_1	z	ν_T	$\nu(u_0)$	$\nu(u_1)$	ν_0	$\bar{\nu}$	λ_{inf}	λ_{sup}	λ_{t+1}
0	0	0	0	0	0	1	1	1	0	0	1	0.5
1	0.5	1.2	2.4	3.6	0.5	1	0.8	1	0.5	0.5	-	0.75
2	0.75	1.4	2.8	4.9	0.75	1	0.6	0.55	0.55	-	0.75	0.62
3	0.62	1.3	2.6	4.22	0.62	1	0.7	0.89	0.62	0.62	-	0.685
4	0.685	1.35	2.7	4.54	0.685	1	0.65	0.73	0.65	0.685	-	0.717
5	0.717	1.37	2.75	4.72	0.717	1	0.62	0.64	0.62	0.717	-	0.733
6	0.733	1.39	2.77	4.8	0.733	1	0.61	0.6	0.61

The optimal satisfaction degree is 0.733, clearly a good compromise solution. Naturally, lower satisfaction degrees become more and more unacceptable; for very low values, the compromise solution can become completely unsatisfactory.

6 Conclusion

This paper proposes a bisection algorithm for the solution of a fuzzy optimal control problem. The optimal solution can be obtained by the solution of a parametric crisp sub-problem.

The convergence of the algorithm is assured in a finite number of steps. The algorithm is extended to *minimum energy* fuzzy quadratic optimal control problem with linear constraints. A numerical test showed satisfactory convergence to the optimal solution.

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